Very basic intro to Intersection numbers

X irreducible variety, dim n

$D_1, \dots, D_k$ Cartier divisors, $k \le n$.

$V \subseteq X$ irreducible subvariety, dim k.

Want: the intersection # $(D_1 \cdot D_2 \cdot \dots \cdot D_k \cdot V) \in \mathbb{Z}$

$$\overset{\shortparallel}{\int_V D_1 \cdot \dots \cdot D_k}$$

Topological description: Each l.b. $\mathcal{O}(D_i)$ has chern class

$$c_1(\mathcal{O}(D_i)) \in H^2(X; \mathbb{Z})$$

cup product $= c_1(\mathcal{O}(D_1)) \cdot \dots \cdot c_1(\mathcal{O}(D_k)) \in H^{2k}(X; \mathbb{Z})$

$[V] \in H_{2k}(X; \mathbb{Z})$

Then $\int_V D_1 \cdot \dots \cdot D_k = (c_1(\mathcal{O}(D_1)) \cdot \dots \cdot c_1(\mathcal{O}(D_k))) \underset{\substack{\uparrow \\ \text{cap product}}}{\cap} [V]$

Properties of intersection # (these characterize it in proj. case)

1.) $\int_X D_1 \cdot \dots D_n$ is symmetric and multilinear on the D_is

2.) $\int_X (D_1 \cdot \dots D_n)$ invariant under linear equivalence of the D_i's.

3.) If $D_1, \dots, D_n$ are effective and meet transversely at smooth points of X, then $\int_X (D_1 \cdots D_n) = \#\{D_1 \cap \cdots \cap D_n\}$

Can find $\int_V D_1 \cdots D_k$ by replacing each D_i w/ lin equiv. $D_i' \not\supset V$ and intersecting restrictions.

Basics of chern class calculations

Goal: be able to do basic intersection computations using Chern classes

X a complete (projective) scheme, E a vector bundle, then
$c_k(E) \in H^{2k}(X;\mathbb{Z})$, $c(E) = 1 + c_1(E) + c_2(E) + \cdots$ the total Chern class,
(or $c_t(E) = 1 + c_1(E)t + c_2(E)t^2 + \cdots$ the Chern polynomial)

Properties

1.) $c_0(E) = 1$

2.) $c_i(E) = 0$ for $i > \operatorname{rank}(E)$

3.) $0 \to E' \to E \to E'' \to 0$ exact seq. of l.b., then $c_t(E) = c_t(E')c_t(E'')$
i.e. $c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$

4.) L, L' l.b. Then $c_1(L \otimes L') = c_1(L) + c_1(L')$

5.) $f: X \to Y$ morphism, then $f^*(c_k(E)) = c_k(f^*(E))$

6.) If X smooth, $\operatorname{rk} E = n$, and σ a generic section, then $c_n(E)$ is the Poincaré dual of $[Z]$, where Z is the zero locus of σ

Note: can define for sheaves more generally (1.) and 3.) hold)

Ex: Use properties of chern classes to show a sm. cubic in $\mathbb{P}^3$ has at most 27 lines. (Work on $\mathbb{G}(1,3)$ and find a relevant v.b.)

Ex: Calculate chern classes of tangent bundle to $\mathbb{P}^n$

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Fujita's conjectures:

Def: X sm. proj. variety of dim n. L a l.b. on X
1.) L is nef if $L \cdot C \geq 0$ for every irred. curve $C \subset X$
2.) If L is nef, then it's big if $L^n > 0$.
(more about big line bundles later)

Conj (Fujita) X sm. projective variety of dimension n
a.) Suppose K_X is big and nef. Then $\mathcal{O}(mK_X)$ is globally generated for $m \geq n+2$.
b.) L an ample l.b. on X. Then $K_X + (n+1)L$ is globally generated and $K_X + (n+2)L$ is very ample.

Holds for curves, surfaces (Reider's Thm), and effective results exist in all dims.

Goal: Prove Reider's Thm using vector bundle techniques.

(Encode l.b. data into v.b. and then study the geom of v.b.)